COVENANT UNIVERSITY NIGERIA

TUTORIAL KIT OMEGA SEMESTER

PROGRAMME: PHYSICS

COURSE: PHY 425

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PHY 425: Fundamentals of Geophysical Data Processing

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1. Explain the following: Data processing, sampling and inversion.
2. State the important of Z-transforms in data analysis.
3. Explain the following: wavelet basis and scaling functions.
4. State the principle of linear superposition; although the subsurface is largely heterogeneous, it obeys the principle of linear superposition.
5. Explain why non-linearity is commonly observed in geophysical data.
6. Given that $Y(Z) = X(Z)B(Z)$, show that the coefficients of the output may be expressed as a convolution equation given by $y_{j+i} = \sum_{i=0}^{n} x_j b_i$.
7. What is meant by realizable and non-realizable filters?
8. Let $B(Z) = 1 + Z + Z^2 + Z^3 + Z^4$, sketch the graph of the coefficients of $B(Z)$ as a function of the powers of Z, compare it with the graph of the coefficients of $[B(Z)]^2$
9. Suppose a wavelet is made up of a complex numbers, determine whether or not the autocorrelation relation $s_k = s_{-k}$ is true, is $s(\omega)$ real or complex?
10. Show that multiplication by $(1 - Z)$ in discretized time is analogous to time differentiation in continuous time and that dividing by $(1 - Z)$ is analogous to integration. What are the limits of the integral?
11. Show that the sum of filters $A(Z) + G(Z)$ is minimum phase.
12. Verify that $\overline{P}(1/Z)P(Z) = 1$ for the general form of an all-pass filter $P(Z) = \frac{Z^N \overline{A}(1/Z)}{A(Z)}$
13. State and explain the reasons why the estimated model in geophysical inverse problems differs from the true model.
14. State the existence and uniqueness theorems of inverse solutions to linear system of equations.
15. Distinguish between sampling theorem and convolution theorem
16. Explain the following: sampling and aliasing, nyquist frequency, and phase and group delay.
17. Distinguish between cross-correlation and auto-correlation.

18. Show that the coefficients of the spectrum of auto-correlation may be expressed as $s_k = \sum_i \bar{b}_i b_{i+k}$; and hence, show that $s_k = s_{-k}$.
19. What are the criteria that a function must be satisfy for it to be qualified as a wavelet.
20. Compare and contrast Fourier transform analysis and wavelet analysis; highlights their strengths and weaknesses.

SOLUTION

1) **Data Processing**: Data processing is the use of computer for the processing and analysis of data. Geophysical involves making inferences of the subsurface from surface or/and near-surface measurements through systematic application of the laws of Physics and the principles of Statistics. Large data set of geophysical often collected requires both statistical reduction and the ability to compute theoretical solution in many parameter earth models.

Sampling: Time and space are ordinarily considered as continuous; however, they have to be discretized for the purpose of computer analysis. This discretizing is also known as digitizing or sampling. Thus, sampling transforms a continuous data form to a discrete data form.

3) **Wavelet Basis**: A wavelet basis is a set of linearly independent functions with a property similar to frequency that is called "detail level" or "scale" but they are also localized in time. A wavelet basis contains two types of functions, scaling functions and wavelets, and each class contains self similar functions.

Scaling Functions: Scaling functions only exist for one detail level (the lowest detail level l_0). Hence, scaling functions describe the basic rough features of a signal while wavelets describe the small variations from the rough average, that add in the details.

5) The plague of *non-linearity* does not arise from geometrical complexities but are often due to large amplitude disturbances. Non-linearity is a dominating feature in hydrodynamics, where flow velocities are a noticeable fraction of the wave velocity.

7) **Realizable Filters**: All physical systems share the property that they do not respond before they are excited. Thus, the impulse response of any physical system is a one-sided time function

that vanishes before $t=0$. Such filters are called realizable or causal filters and respond at or after the lag-time $t=0$.

Non-Realizable Filters: If a filter (or inverse filter) responds to future inputs, then the filter (or inverse filter) is not physically realizable and are referred to as non-realizable (non-causal) filters. However, non-realizable filters are very useful in simulation or modelling for prediction.

9) The spectrum of a signal may be defined as the magnitude squared of the Fourier transform of the function. Let the spectrum be denoted as $S(\omega)$, where $S(\omega)=|B(\omega)|^2=\overline{B}(\omega)B(\omega)$. By expressing this in terms of a three-point Z-transform, we obtained

$$S(\omega)=\left(\overline{b}_0+\overline{b}_1e^{-i\omega}+\overline{b}_2e^{-i2\omega}\right)\left(b_0+b_1e^{i\omega}+b_2e^{i2\omega}\right)$$

$$S(Z)=\left(\overline{b}_0+\frac{\overline{b}_1}{Z}+\frac{\overline{b}_2}{Z^2}\right)\left(b_0+b_1Z+b_2Z^2\right)$$

$$S(Z)=\overline{B}\left(\frac{1}{Z}\right)B(Z)$$

The coefficients of $S(Z)$ can be examined by multiplying out the polynomial $\overline{B}(1/Z)$ with $B(Z)$ so that

$$S(Z)=s_{-2}Z^{-2}+s_{-1}Z^{-1}+s_0+s_1Z+s_2Z^2$$

In general, the coefficient s_k of Z^k is given as

$$s_k=\sum_i \overline{b}_i b_{i+k}$$

This equation is known as the autocorrelation formula. The autocorrelation value s_k at lag 10 (s_{10}) is a measure of the similarity of b_i with itself shifted 10 units in time. In most cases, b_i and by inspection, the autocorrelation coefficients are real, and thus, $s_k=s_{-k}$.

11) To determine if the sum of filters $A(Z)+G(Z)$ is minimum phase, let the filter $A(Z)$ be minimum phase, then

$$A(Z)+G(Z)=A(Z)\left[1+\frac{G(Z)}{A(Z)}\right]$$

The question whether $A(Z)+G(Z)$ is minimum phase has been reduced to whether $A(Z)$ and $1+G(Z)/A(Z)$ are both minimum phase. Since $A(Z)$ is causal (realizable) and $1/A(Z)$ is expandable in the positive powers of Z only, then $G(Z)/A(Z)$ is also causal. A sufficient condition for $1+G(Z)/A(Z)$ to be minimum phase is that the spectrum of $A(Z)$ should exceed that of $G(Z)$ at all frequencies. In other words, for any real ω, $|A|>|G|$. Thus, a plot of $G(Z)/A(Z)$ in the complex plane lies everywhere inside the unit circle and $1+G(Z)/A(Z)$ will always give positive real part.

13) There are two main reasons why the estimated model differs from the true model in the solution to geophysical inverse problems:

i. Non-uniqueness: The non-uniqueness of solution to inverse problem causes several models (usually infinitely many) to fit or explain the observed data set equally well.
ii. Error propagation: The observed data and physical theories are always contaminated with errors and the estimated model is therefore affected by these errors as well.

15) **Sampling Theorem**: Sampling theorem states that no information is lost by regular sampling provided that the sampling frequency is greater than twice the highest frequency component in the waveform being sampled. Thus, there must be more than two samples per cycle for the highest frequency. The sampling theorem determines the minimum sampling we can use. Nothing is gained by using finer sampling since minimum sampling allows complete recovery of the waveform.

Convolution Theorem: Convolution theorem states that the Fourier transform of the convolution of two functions is equal to the product of the transforms of the individual functions.

Thus, if $x_t \leftrightarrow X(f)=|X(f)|e^{i\phi_x(f)}$ and $g_t \leftrightarrow G(f)=|G(f)|e^{i\phi_g(f)}$, then

$$x_t g_t \leftrightarrow X(f)G(f)=[|X(f)|e^{i\phi_x(f)}][|G(f)|e^{i\phi_g(f)}]$$
$$\leftrightarrow |X(f)||G(f)|e^{i[\phi_x(f)+\phi_g(f)]}$$

where $|X(f)|$ and $|G(f)|$ are the amplitude spectra, $\phi_x(f)$ and $\phi_g(f)$ are the phase spectra. Thus, if two sets of data are convolved in the time domain, the effect of frequency domain is to multiply their amplitude spectra and add their phase spectra. Because of certain symmetry properties of Fourier transform, it can be shown that

$$x_t g_t \leftrightarrow X(f) * G(f)$$

17) The cross-correlation function is a measure of the similarity between two data sets. Corresponding values of the two sets are multiplied together and the product summed to give the value of the cross-correlation. Wherever the two sets are nearly the same, the products will usually be positive and hence the cross-correlation is large. Wherever the sets are unlike, some of the products will be positive and some negative and hence the cross-correlation will be small. A large negative value in the cross-correlation function implies that the two sets would be similar if one were inverted. Thus, the two sets are similar but out-of-phase. The cross-correlation of two data sets x_t and y_t can be expressed as

$$\Phi_{xy}(\tau) = \sum_k x_{x+\tau} y_k \; .$$

where τ is the displacement of x_t relative to y_t. Cross-correlation can be performed by reversing the second data set and convolving, that is, $\Phi_{xy}(\tau) = \Phi_{yx}(-\tau)$. Since forming the complex conjugate involves only reversing the sign of the phase, cross-correlation is equivalent to multiplying the amplitude spectra and subtracting the phase spectra. Thus,

$$x_t \leftrightarrow X(f) = |X(f)| e^{i\phi_x}$$

$$y_t \leftrightarrow Y(f) = |Y(f)| e^{i\phi_y}$$

$$y_{-t} \leftrightarrow \overline{Y}(f) = |Y(f)| e^{-i\phi_y}$$

$$\Phi_{xy}(\tau) = X(f)\overline{Y}(f) = |X(f)||Y(f)| e^{i(\phi_x - \phi_y)}$$

Changing the sign of a phase spectrum is equivalent to reversing the trace in time domain.

Autocorrelation is a special case of cross-correlation where a data set is correlated with itself, that is $\Phi_{xx}(\tau) = \sum_k x_{k+\tau} x_k$. Autocorrelation functions are symmetrical because a time shift to the right is the same as a shift to the left $\Phi_{xx}(\tau) = \Phi_{xx}(-\tau)$. Autocorrelation has its peak value at zero time shift, that is, a data set is most like itself before it is time-shifted. If the autocorrelation have large a value at some time $\Delta t \neq 0$, it indicates that the set tends to be periodic with the period Δt. Thus, autocorrelation function may be thought of as the repetitiveness of a function.

19) For a function $\psi(t)$ to be classified as wavelet, it must satisfy the following mathematical criteria:

1. The wavelet $\psi(t)$ must have finite energy given by:

$$E = \int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty$$

2. If $\Psi(f)$ is the Fourier transform of $\psi(t)$, that is

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(t) e^{-i2\pi ft}\, dt$$

   Then, the following condition must hold:

$$C_g = \int_0^{\infty} \frac{|\Psi(f)|^2}{f} df < \infty$$

   This implies that the wavelet has no zero frequency component, $\Psi(0) = 0$, that is, the wavelet $\psi(t)$ must have a zero mean. This relation is known as the admissibility condition and C_g is called the admissibility constant.

3. An additional criterion that must hold for complex wavelets is that the Fourier transform must both be real and vanish for negative frequencies.