Modified Burr III Distribution: Ordinary Differential Equations

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Abstract— Modified Burr III distribution is an improved model obtained from the modification of Burr III distribution. In this work, differentiation was applied to obtain the ordinary differential equations (ODE) of the probability functions of modified Burr III distribution. The parameters and support that characterized the distribution inevitably determine the nature, existence, uniqueness and solution of the ODEs. The method is recommended to be applied to other probability distributions and probability functions not considered in this paper. Computer codes and programs can be used for the implementation.

Index Terms— Differential calculus, quantile function, hazard function, reversed hazard function, survival function, inverse survival function, probability density function, Burr.

I. INTRODUCTION

CALCULUS in general and differential calculus in particular is often used in statistics in parameter and modal estimations. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-4].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of distributions can be transformed as ODE whose solution yields the respective PDF. Some of which are available: see [5-9].

The aim of this paper is to develop homogenous ODEs for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the modified Burr III distribution. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the conditions necessary for the existence of the ODEs. Similar results for other distributions have been proposed, see [10-23] for details.

Ali et al. [24] proposed the distribution as an improvement over the parent (Burr III) distribution. Differential calculus was used to obtain the results.

II. PROBABILITY DENSITY FUNCTION

The PDF of the modified Burr III distribution is given as:

\[ f(x) = \alpha \beta x^{-(\beta + 1)} \left[ 1 + \gamma x^{-\beta} \right] \left( \frac{\alpha + 1}{\gamma} \right) \]

Differentiate equation (1);

\[ f'(x) = \frac{(\beta + 1) x^{-(\beta + 2)}}{x^{-(\beta + 1)}} + (\alpha + \gamma) x^{-\beta} \left[ 1 + \gamma x^{-\beta} \right] \left( \frac{\alpha + 1}{\gamma} \right) f(x) \]

\[ f'(x) = \frac{(\beta + 1)}{x} + \frac{(\alpha + \gamma) x^{-(\beta + 1)}}{1 + \gamma x^{-\beta}} f(x) \]

The equation can only exists for \( x, \alpha, \beta, \gamma > 0 \).

The first order ODE for the PDF of modified Burr III distribution can be obtained from any given values of \( \alpha, \beta, \gamma \). Some examples are given in Table 1.

Table 1: ODE for the PDF for some given values of the parameters

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \alpha )</th>
<th>Ordinary Differential Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( (x+1)f'(x) + 2f(x) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( x(x+1)f'(x) + (2x-1)f(x) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>( x(x+2)f'(x) + (2x+1)f(x) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>( (x+2)f'(x) + 2f(x) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>( x(x^2+1)f'(x) + (3x^2-1)f(x) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>( x(x^2+1)f'(x) + 3(x^2-1)f(x) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>( x(x^2+1)f'(x) + 3(x^2-1)f(x) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( x(x^2+1)f'(x) + (3x^2-5)f(x) = 0 )</td>
</tr>
</tbody>
</table>
Differentiate equation (3);

\[
f''(x) = \left\{ \frac{(\beta+1) + (\alpha + \gamma)\gamma\beta^2(x^{-\beta+1})^2}{x^2(1+\gamma x^{-\beta})^2} - \frac{\beta(\beta+1)(\alpha+\gamma)x^{-\beta+2}}{(1+\gamma x^{-\beta})} \right\} f'(x) + \left\{ \frac{(\beta+1)}{x} + \frac{(\alpha + \gamma)\beta x^{-(\beta+1)}}{1+\gamma x^{-\beta}} \right\} f'(x) \tag{4}
\]

The equation can only exists for \( x, \alpha, \beta, \gamma > 0 \).

The presented equations acquired from equation (3) are required in the further breaking down of equation (4);

\[
\frac{-(\beta+1)}{x} + \frac{(\alpha + \gamma)\beta x^{-(\beta+1)}}{1+\gamma x^{-\beta}} = f''(x) \tag{5}
\]

\[
\frac{(\alpha + \gamma)\gamma\beta^2(x^{-\beta+1})^2}{(1+\gamma x^{-\beta})^2} = f''(x) + \frac{(\beta+1)}{f(x)} \tag{6}
\]

\[
\left( \frac{\alpha + \gamma}{1+\gamma x^{-\beta}} \right)^2 = \left( \frac{f''(x) + \frac{\beta+1}{x}}{f(x)} \right)^2 \tag{7}
\]

\[
\frac{(\alpha + \gamma)\gamma^2\beta^2(x^{-\beta+1})^2}{(1+\gamma x^{-\beta})^2} = \frac{\gamma}{\alpha + \gamma} \left( \frac{f''(x) + \frac{\beta+1}{x}}{f(x)} \right)^2 \tag{8}
\]

\[
\frac{(\alpha + \gamma)\beta(\beta+1)x^{-(\beta+1)}}{1+\gamma x^{-\beta}} = \beta+1 \left( \frac{f''(x) + \frac{(\beta+1)}{x}}{f(x)} \right) \tag{9}
\]

\[
\frac{(\alpha + \gamma)\beta(\beta+1)x^{-(\beta+2)}}{1+\gamma x^{-\beta}} = \beta+1 \left( \frac{f''(x) + \frac{(\beta+1)}{x}}{f(x)} \right) \tag{10}
\]

Substitute equations (5), (8) and (10) into equation (4);

\[
f''(x) = f''(x) + \frac{\beta+1}{x} \left( \frac{f''(x) + \frac{(\beta+1)}{x}}{f(x)} \right) \tag{11}
\]

\[
f(1) = \alpha\beta(1+\gamma) \tag{12}
\]

\[
f'(1) = \left[ \frac{(\alpha + \gamma)(\beta+1)(1+\gamma)}{1+\gamma} \right] \tag{13}
\]

The ODE can be obtained for \( \alpha, \beta, \gamma \).

The results are similar to the ones proposed in [10-23].

III. QUANTILE FUNCTION

The QF of the Modified Burr III distribution is given as;

\[
Q(p) = \left( \frac{1}{\gamma} \left[ \frac{1}{p^\alpha} - 1 \right] \right)^{-\frac{1}{\beta}} \tag{14}
\]

Differentiate equation (14), to obtain the first order ODE;

\[
Q'(p) = \frac{p^{-\frac{1}{\alpha}}}{\alpha \beta} \left( \frac{1}{\gamma} \left[ \frac{1}{p^\alpha} - 1 \right] \right)^{-\frac{1}{\beta}} \tag{15}
\]

\[
Q'(p) = \frac{\alpha \beta p^{-\frac{1}{\alpha}} \left[ \frac{1}{\gamma} \left[ \frac{1}{p^\alpha} - 1 \right] \right]}{\alpha \beta p^{-\frac{1}{\alpha}} \left[ \frac{1}{\gamma} \left[ \frac{1}{p^\alpha} - 1 \right] \right]} \tag{16}
\]

The equation can only exists for \( \alpha, \beta, \gamma > 0, 0 < p < 1 \).

Substitute equation (14);

\[
Q'(p) = \frac{\gamma Q(p)}{\alpha \beta p^{-\frac{1}{\alpha}} \left[ \frac{1}{\gamma} \left[ \frac{1}{p^\alpha} - 1 \right] \right]} \tag{17}
\]

\[
Q'(p) = \frac{\gamma Q(p)}{\alpha \beta (p-\frac{1}{p^\alpha})} \tag{18}
\]

\[
Q'(p) = \frac{\gamma Q(p)}{\alpha \beta p(1-p^{-\frac{1}{\alpha}})} \tag{19}
\]

\[
\alpha \beta p(1-p^{-\frac{1}{\alpha}})Q'(p) - \gamma Q(p) = 0 \tag{20}
\]

The first order ODE for the QF of the modified Burr III distribution can be acquired for the particular values of \( \alpha, \beta, \gamma \). Some examples are given in Table 2.
Table 2: ODE for the QF for some selected values

<table>
<thead>
<tr>
<th>γ</th>
<th>α</th>
<th>β</th>
<th>Ordinary Differential Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( p(1-p)Q'(p) - Q(p) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( 2p(1-p)Q'(p) - Q(p) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>( p(1-p^2)Q'(p) - 2Q(p) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>( p(1-p^2)Q'(p) - Q(p) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>( p(1-p)Q'(p) - Q(p) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( 2p(1-p)Q'(p) - Q(p) = 0 )</td>
</tr>
</tbody>
</table>

The results are similar to the ones proposed in [10-23].

IV. SURVIVAL FUNCTION

The SF of the Modified Burr III distribution is given as:

\[
S(t) = 1 - [1 + \gamma t^{-\beta}]^{-\frac{\alpha}{\gamma}}
\]

(21)

Differentiate equation (21), to obtain the first order ODE:

\[
S'(t) = -\alpha \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{-\frac{\alpha}{\gamma}}
\]

(22)

\[
S'(t) = -\frac{\alpha \beta(1-S(t))}{\gamma t^{\beta+1} + \gamma t}
\]

(23)

The equation can only exists for \( t, \alpha, \beta, \gamma > 0 \).

Substitute equation (21) into (23);

\[
S'(t) = -\frac{\alpha \beta(1-S(t))}{\gamma t^{\beta+1} + \gamma t}
\]

(24)

\[
S'(t) = -\frac{\alpha \beta(1-S(t))}{\gamma t^{\beta+1} + \gamma t}
\]

(25)

\[
S'(t) = -\frac{\alpha \beta(1-S(t))}{\gamma t^{\beta+1} + \gamma t}
\]

(26)

The first order ODE for the SF of the modified Burr III distribution can be obtained for any given values of \( \alpha, \beta, \gamma \). Some examples are given in Table 3.

Table 3: ODE for the SF for some selected values

<table>
<thead>
<tr>
<th>β</th>
<th>γ</th>
<th>α</th>
<th>Ordinary Differential Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( t(t+1)S'(t) - S(t) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( t(t+1)S'(t) - 2S(t) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>( t(t+2)S'(t) - S(t) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>( t(t+2)S'(t) - 2S(t) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>( t(t+1)S'(t) - 2S(t) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>( t(t+1)S'(t) - 4S(t) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>( t(t+2)S'(t) - 2S(t) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( t(t+2)S'(t) - 4S(t) = 0 )</td>
</tr>
</tbody>
</table>

Table 4: ODE for the SF for some selected values

<table>
<thead>
<tr>
<th>γ</th>
<th>α</th>
<th>β</th>
<th>Ordinary Differential Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( p(1-p)Q'(p) + Q(p) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( 2p(1-p)Q'(p) + Q(p) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>( 2(1-p)(1-\sqrt{p})Q'(p) + Q(p) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>( 4(1-p)(1-\sqrt{p})Q'(p) + Q(p) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>( (1-p)(1-p^{2})Q'(p) + 2Q(p) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>( (1-p)(1-p^{2})Q'(p) + Q(p) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>( p(1-p)Q'(p) + Q(p) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( 2p(1-p)Q'(p) + Q(p) = 0 )</td>
</tr>
</tbody>
</table>

The results are similar to the ones proposed in [10-23].

V. INVERSE SURVIVAL FUNCTION

The ISF of the modified Burr III distribution is given as:

\[
Q(p) = \left(\frac{1}{\gamma} \left[\frac{1}{(1-p)^{\frac{1}{\alpha}}} - 1\right]\right)^{\frac{1}{\beta}}
\]

(28)

Differentiate equation (28) to obtain the first order ODE:

\[
Q'(p) = -\frac{(1-p)^{\frac{\beta+1}{\alpha}}}{\alpha \beta} \left(\frac{1}{\gamma} \left[\frac{1}{(1-p)^{\frac{1}{\alpha}}} - 1\right]\right)^{\frac{1}{\beta} - 1}
\]

(29)

The equation can only exists for \( \alpha, \beta, \gamma > 0, 0 < p < 1 \).

Substitute equation (28) into equation (30);

\[
Q'(p) = -\frac{\gamma Q(p)}{\alpha \beta(1-p)^{\frac{\beta+1}{\alpha}}} \left(\frac{1}{\gamma} \left[\frac{1}{(1-p)^{\frac{1}{\alpha}}} - 1\right]\right)^{\frac{1}{\beta} - 1}
\]

(31)

\[
Q'(p) = -\frac{\gamma Q(p)}{\alpha \beta(1-p)^{\frac{\beta+1}{\alpha}}} \left(\frac{1}{\gamma} \left[\frac{1}{(1-p)^{\frac{1}{\alpha}}} - 1\right]\right)^{\frac{1}{\beta} - 1}
\]

(32)

\[
Q'(p) = -\frac{\gamma Q(p)}{\alpha \beta(1-p)^{\frac{\beta+1}{\alpha}}} \left(\frac{1}{\gamma} \left[\frac{1}{(1-p)^{\frac{1}{\alpha}}} - 1\right]\right)^{\frac{1}{\beta} - 1}
\]

(33)

\[
\alpha \beta(1-p)(1-(1-p)^{\frac{\beta+1}{\alpha}})Q'(p) + \gamma Q(p) = 0
\]

(34)

The first order ODE for the ISF of the Modified Burr III distribution can be obtained for any given values of \( \alpha, \beta, \gamma \). Some examples are given in Table 4.
The results are similar to the ones proposed in [10-23].

VI. HAZARD FUNCTION

The HF of the Modified Burr III distribution is given as;

\[
h(t) = \frac{\alpha \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}
\]

(35)

Differentiate equation (35) to obtain the first order ODE;

\[
h'(t) = \frac{-t^{-(\beta+1)} + (\alpha + \gamma) \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}
\]

(36)

The equation can only exists for \( t, \alpha, \beta, \gamma > 0 \).

\[
h'(t) = \frac{-(\beta + 1) t^{-(\beta+1)}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}} + \frac{(\alpha + \gamma) \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}} + h(t)
\]

(37)

The first order ODE for the HF of the Modified Burr III distribution can be obtained for any given values of \( \alpha, \beta, \gamma \). Some examples are given in Table 5.

Table 5: ODE for the HF for some selected values

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \alpha )</th>
<th>Ordinary Differential Equations</th>
</tr>
</thead>
</table>
| 1 | 1 | 1 | \( (t+1)h'(t) + 2h(t) \)  
\(-t(t+1)h'(t) = 0 \)
| 1 | 1 | 2 | \( t(t+1)h'(t) + (2t-1)h(t) \)  
\(-t(t+1)h'(t) = 0 \)

The results are similar to the ones proposed in [10-23]. Differentiate equation (38);

\[
h''(t) = \left\{ \frac{h'(t) + \frac{\beta + 1}{t^2} + (\alpha + \gamma) \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}} + h(t) \right\} h'(t)
\]

(39)

The equation can only exists for \( t, \alpha, \beta, \gamma > 0 \).

The presented equations derived from equation (38) are required in the further breaking down of equation (39);

\[
\frac{-(\beta + 1)}{t} + (\alpha + \gamma) \beta t^{-(\beta+1)} + h(t) = h'(t)
\]

(40)

\[
\frac{(\alpha + \gamma) \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}} = \frac{h'(t)}{h(t)} + \frac{\beta + 1}{t} - h(t)
\]

(41)

\[
\frac{(\alpha + \gamma) \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}} = \left( \frac{h'(t)}{h(t)} + \frac{\beta + 1}{t} - h(t) \right)^2
\]

(42)

\[
\frac{(\alpha + \gamma) \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}} = \frac{\gamma}{\alpha + \gamma} \left( \frac{h'(t)}{h(t)} + \frac{\beta + 1}{t} - h(t) \right)
\]

(43)

\[
\frac{(\alpha + \gamma) \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}} = \beta + 1 \left( \frac{h'(t)}{h(t)} + \frac{\beta + 1}{t} - h(t) \right)
\]

(44)

\[
\frac{(\alpha + \gamma) \beta t^{-(\beta+1)} \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}}{1 - \left[1 + \gamma t^{-\beta}\right]^{\alpha+1}} = \beta + 1 \left( \frac{h'(t)}{h(t)} + \frac{\beta + 1}{t} - h(t) \right)
\]

(45)

Substitute equations (40), (43) and (45) into equation (39);
\[ h(1) = \frac{\alpha \beta [1 + \gamma (t + 1)]^{\frac{2}{\gamma}}}{1 - [1 + \gamma]^{\frac{2}{\gamma}}} \]  
(47)

\[ h'(1) = \left\{ \frac{(\beta + 1) t^{-\beta - 1} + (\alpha + \gamma) \beta}{1 + \gamma} + h(1) \right\} h(1) \]  
(48)

The ODEs can be obtained for any given values of \( \alpha, \beta, \gamma \).

VII. REVERSED HAZARD FUNCTION

The RHF of the modified Burr III distribution is given as;

\[ j(t) = \frac{\alpha \beta t^{-(\beta + 1)}}{1 + \gamma t^\beta} \]  
(49)

Differentiate equation (49), to obtain the first order ODE;

\[ j'(t) = \left\{ \frac{(\beta + 1) t^{-(\beta + 2)}}{t^{-(\beta + 1)}} + \frac{\beta \gamma t^{-(\beta + 1)}(1 + \gamma t^{-\beta})^{-2}}{(1 + \gamma t^{-\beta})^{-1}} \right\} j(t) \]  
(50)

\[ j'(t) = \left\{ \frac{(\beta + 1) t^{-\beta}}{1 + \gamma t^{-\beta}} \right\} j(t) \]  
(51)

The equation can only exists for \( t, \alpha, \beta, \gamma > 0 \).

\[ j'(t) = \frac{(\beta + 1) t}{\alpha} j(t) \]  
(52)

The first order ODE for the RHF of the modified Burr III distribution is given by;

\[ \alpha j'(t) + \alpha (\beta + 1) j(t) - t^2 j(t) = 0 \]  
(53)

\[ j(1) = \frac{\alpha \beta}{1 + \gamma} \]  
(54)

The results are similar to the ones proposed in [10-23].

VIII. CONCLUDING REMARKS

Ordinary differential equations (ODEs) have been obtained for the probability functions of modified Burr III distribution. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. The parameter and the supports that characterize the distribution determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs. [25-40]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

ACKNOWLEDGMENT

The comments of the reviewers were very helpful and led to an improvement of the paper. This research benefited from sponsorship from the Statistics sub-cluster of the Industrial Mathematics Research Group (TIMREG) of Covenant University and Centre for Research, Innovation and Discovery (CURID), Covenant University, Ota, Nigeria.

REFERENCES


