Abstract—This paper addresses the existence of solution of Impulsive Quantum Stochastic Differential Inclusion (also known as impulsive non-classical ordinary differential inclusion) with an additional bounded linear operator. The multivalued maps are necessarily compact. By showing that the multivalued maps can actually satisfy some conditions of appropriate fixed-point theorems, the major result is established.

Index Terms—QSDI, Stochastic processes, Bounded Linear Operator, Fixed point.

I. INTRODUCTION

Some interesting results concerning the following quantum stochastic differential equation (QSDE) and inclusion (QSDI) introduced by Hudson and Parthasarathy [12]:

\[
dy(t) = E(t, y(t))d\Delta_x(t) + F(t, y(t))d\Lambda(t) + G(t, y(t))dA^+_T(t) + h(t, y(t))dt.
\]

\[
y(0) = y_0
\]

have been studied. See [1, 4-6, 10]. However, It has shown by Ekhaguere [10], that the following nonclassical ordinary differential inclusion (NODI):

\[
\frac{d}{dt}\langle \eta, y(t) \rangle = P(t, y)(\eta, \xi)
\]

\[
\langle \eta, y(0) \rangle = \langle \eta, y_0 \rangle
\]

is equivalent to inclusion (1) where the map P in (2) is well defined in [1]. The definitions and notations of some spaces such as \(L_2^\infty([0, T], \mathbb{R}^n)\), \(L^\infty([0, T], \mathbb{R}^n)\), \(L^p([0, T], \mathbb{R}^n)\), \(\Gamma((L^2_0([0, T], \mathbb{R}^n)))\) are adopted from the reference [6] and the references therein.

(i) The stochastic processes are densely defined linear operators on the complex Hilbert space \(\mathcal{R} \otimes \Gamma(L^2_0([0, T], \mathbb{R}^n))\). We denote by \(\langle \cdot, \cdot \rangle\) the inner product of \(\mathcal{R} \otimes \Gamma(L^2_0([0, T], \mathbb{R}^n))\) and \(\| \cdot \|\) the norm induced by \(\langle \cdot, \cdot \rangle\).

(ii) \(U, V, W, H, \) are coefficients which are in \(L^2_{loc}(\mathcal{A} \times [0, T])\), \(\mathcal{A}\) is a locally convex space. The elements of \(\mathcal{A}\) consists of linear maps from \(\mathcal{D} \otimes \mathcal{E}\) into \(\mathcal{R} \otimes \Gamma(L^2_0([0, T], \mathbb{R}^n))\). For the definitions of the integrators \(A^+_T, A^+_T, A^+_T, A^+_T\) in (1.1), see the references [1, 8, 10].

(iii) \(\mathcal{D}\) is an inner product space, \(\mathcal{R}\) is the completion of \(\mathcal{D}\) and \(\mathcal{E}\) a linear space in \(\Gamma(L^2_0([0, T], \mathbb{R}^n))\).

Differential inclusions with impulsive effects have found more practical applications than differential equations without impulse effect [2-3, 9, 11], hence the need to extend some of these results to this noncommutative quantum setting. For more on impulsive ordinary differential inclusions and their applications, see the references [2-4]. Bishop et. al. [4] studied results on impulsive (QSDE) with nonlocal conditions. The method used was simply an adoption of the methods employed in the classical setting. Similar results are established in [6] with strict conditions on the maps. In this paper, the conditions on the maps are weakened with an additional bounded linear operator.

To the best of our knowledge within the consulted literature, there are no results concerning impulsive QSDI in the sense of this paper. However, in the classical setting, several results on differential equations/inclusions with impulse effects have been established. See the references [2, 3] and the references therein. This work generalizes existing results in the literature on impulsive quantum stochastic differential equations since some of these results can be recovered from the result in this paper. The remaining part of this paper will consist of two sections: Section 2 will contain some preliminaries which we adopt from the references [1, 4, 5, 10], while in section 3, the main result will be established.

In this paper, the fixed-point method is used to establish the major result. In recent times, this method has proven to be very efficient in establishing existence of solutions of differential equations and approximate solutions.

II. PRELIMINARIES

Let \(\mathcal{A}\) be a topological space and \(\text{clos}(\mathcal{A}), \text{comp}(\mathcal{A})\) be as defined in [6]. Define \(I = [0, b] \), \(I' = I \setminus \{t_1, t_2, \ldots, t_m\}\), \(0 < t_1 < \ldots < t_m < t_{m+1} = b\).

Definitions 2.1

(i) \(PC(I, \mathcal{A}) = \{y : I \rightarrow \mathcal{A} : y(t)\) is continuous and \(y(t_k) = y(t_1)\)\)
We define the sesquilinear equivalent forms $PC(I, A(\eta, \xi))$ associated with $PC(I, \text{sesq}(A \otimes \Xi))$ in the same way with the above, where $A$ is a subset of $\mathbb{C}$. (ii) $PC(I, A(\eta, \xi))$ is a Banach space whose norm is defined by $||y||_{R^t, PC} = \sup \{ |y(t)| : t \in [0, T] \}$, where $A \subseteq \mathbb{C}$.

(iii) $\rho(B, C) := \max \{ \delta(B, C), \delta(C, B) \}$, $B, C \in \text{clo}_{\mathbb{C}}(C)$ and $d(y, B) = \inf_{y \in B} |x - y|$, where $y \in \mathbb{C}$, is a complex number, $\rho(B, C)$ the Hausdorff distance, and $\rho$ a metric on $\text{clo}_{\mathbb{C}}(C)$.

Note: (i) in definition 2.1 holds everywhere except for some $t_k$.

**Definition 2.2**
For the rest of this paper, $\Phi$ indexed by $I = [0, T] \subseteq \mathbb{R}$, is a multivalued stochastic process.

(i) $\Phi$ is $L^1$-measurable in the sense of [2]

(ii) A selection of $\Phi$ is a stochastic process $y : I \to A$ such that $y(t) \in \Phi(t)$ for almost all $t \in I$.

(iii) $\Phi$ is said to be lower semicontinuous if for every open set $V \subseteq A$, $\Phi^-(V)$ is open.

(iv) $\Phi$ is adapted if $\Phi(t) \subseteq A_t$, $t \in I$ and its denoted by $\text{Ad}(A)$.

(v) A member $y \in \text{Ad}(A)$ is said to be weakly absolutely continuous (AC) if $t \to (\eta, y(t) \xi) : t \in I$ is AC for $\eta, \xi \in D \otimes \Xi$. Denote all such sets by $\text{Ad}(A)_{\text{AC}}$.

(vi) If $t \to \|y(t)\|_R \in L^p_{\text{loc}}(A), p \in (0, \infty)$, then $y(t)$ is locally absolutely $p$-integrable.

Denote this set by $L^p_{\text{loc}}(A)_{\text{AC}}$.

Also define the set $S^\Phi$ by

$$S^\Phi = \{ g \in L^1(\mathbb{A}, I), g(t) \in \Phi(y(t), t), \ a.e. \ t \in I \} \neq \emptyset$$

Note: Subsequently, $\eta, \xi \in D \otimes \Xi$ is arbitrary.

Next, we introduce the following QSDI with an additional bounded linear operator:

$$dy(t) - A(t)y(t) \in B(y(t)) + U'(y(t), t)d\epsilon(t) + V(y(t))dA_y(t) + W(y(t), t)dA^2_y(t) + H(y(t), t),$$

$$\Delta y(t) = J_k(y(t_k)), t = t_k, k = 1, \ldots, m$$

(3)

Observe that, the following inclusion is equivalent to inclusion (3).

$$\frac{d}{dt} (\eta, y(t) - A(t) y(t)) \in B(y(t)) + P(t, y) (\eta, \xi)$$

for some $t \in I, t \neq t_k, k = 1, \ldots, m$, 

$$\Delta y(t) = J_k(y(t_k)) (\eta, \xi), t = t_k,$$ 

$$\langle \eta, y(0) \xi \rangle = \langle \eta, y_0 \xi \rangle$$

(4)

where the multivalued map $P : \mathbb{A} \times I \to \text{clo}_{\mathbb{C}}(A)$ is assumed to be compact and convex, $y \to \Phi(y(t), t), \Phi \subseteq \{ A, F, G, H \}$ is a multivalued stochastic process, $y_0 \in \text{clo}_{\mathbb{C}}(A) \text{ and } B : \mathbb{A} \to \mathbb{A}, J_k \in \text{clo}_{\mathbb{C}}(A, \mathbb{A})$.

**Definition 2.3**

(i) $\Delta y(t) = y(t_k^\pm) - y(t_k^-), t = t_k,$

$$y(t_k^\pm) = \lim_{h \to t_k^\pm} y(t_k + h)$$

(ii) $y(t_k^-) = \lim_{k \to t_k^-, h} y(t_k - h), k = 1, \ldots, m$.

(iii) Let $y \in L^1(I, \text{Ad}(A)_{\text{AC}}) \cap PC(I, A)$, then $y(t) \in (\text{Ad}(A)_{\text{AC}})$ is a solution of (3) if it satisfies the following integral inclusion:

$$y(t) \in \left\{ T(t-s) B(y(s))(\eta, \xi) \right\} + \int_0^t T(t-s) B(y(s))(\eta, \xi) ds$$

$$+ \sum_{0 < t_k < t} J_k(y(t_k))(\eta, \xi)$$

(5)

The following conditions are assumed:

(S1) $\Phi : \mathbb{A} \times I \to \text{clo}_{\mathbb{C}}(A)$ are compact and convex such that $y \to \Phi(y(t), t)$ satisfies conditions (iii) and (iv) of Definition 2.2 above.

(S2) $\|\Phi(y(t), t)\|_R \leq c_k, k = 1, \ldots, m$ for each $y \in \mathbb{A}$, where $c_k$ are constants.

(S3) $A$ is the infinitesimal generator of a family of semigroup $\{T(t), t \geq 0\}$ and $T$ is compact for $t > 0$, such that $\|T(t)\|_{\text{R}} \leq M$, where $M$ is a constant, $t \geq 0$. $B$ is an additional bounded linear operator with values in $\mathbb{A}$.

(S4) We define a function $\Psi : [0, \infty) \to (0, \infty)$ which is continuous and nondecreasing such that $\|\Phi(y(t), t)\|_R \leq \Psi(t) \|\Phi(y(t), t)\|_R$ for all $t \in I$ and $\Psi(t)$ is a bounded linear operator with values in $\mathbb{A}$.

(S5) $y(t) \in \text{clo}_{\mathbb{C}}(A) \text{ and } \sup_t ||y(t)\|_R < \infty$.

III. MAIN RESULTS

**Theorem 3.1** Assume conditions (S1) - (S4) hold, then (3) has at least a weak solution.

Proof: Firstly, (4) has to be transformed into a fixed-point problem.

Define $N : PC(I, A(\eta, \xi)) \to \text{sesq}(D \otimes \Xi)$ by $N(y(t), (\eta, \xi)) = (\eta, T(t)y_0 \xi) + \int_0^t T(t-s) B(y(s))(\eta, \xi) ds$

$$+ \sum_{0 < t_k < t} J_k(y(t_k))(\eta, \xi)$$

(8)

Let $K := \{ y \in (I, \text{sesq}(D \otimes \Xi)) : \|y(t)\|_R \leq b(t) \}$ (9)

where $b(t) = J^\perp \left( \int_0^b \Psi(s) ds \right)$

(10)

Hence by (9), the set $K$ is: (a) compact and (b) convex.
Assume $K^* = \sup \{\|y\|_{\mathcal{F}}: y \in K\}.
We show that $N(K) \subset K$ and is also relatively compact.
Fix $t \in I$ and let $g \in \mathbb{S}^\Phi$ such that,
$h(t)(\eta, \xi) = \langle \eta, T(t)y_0 \xi \rangle$
$+ \int_0^t T(t-s)B(y(s))(\eta, \xi) \, ds + \int_0^t T(t-s)g_{\mathcal{H}}(s) \, ds$
$+ \sum_{0 < t_k < t} T(t-s-t_k)J_k(y(t_k))(\eta, \xi)$
(11)
Therefore,
$h(t)(\eta, \xi) = M\|y_0\|_{\mathcal{F}}$
$+ M \sum_{k=1}^m c_k + \int_0^t w^*(s)(\| y(s) \|_{\mathcal{F}} + \Psi(\| y(s) \|_{\mathcal{F}})) \, ds$
$\leq M\|y_0\|_{\mathcal{F}} + M \sum_{k=1}^m c_k + \int_0^t w^*(s)(b(s) + \Psi(b(s))) \, ds$
(12)
$\leq M\|y_0\|_{\mathcal{F}} + M \sum_{k=1}^m c_k + \int_0^t b(s) \, ds = b(s)$
given that
$\int_0^b \frac{du}{u + \Psi(u)} = \int_0^t w(\tau) \, d\tau, \quad d > 0$ (13)
where $h \in PC(I, \mathbb{R}) \subset K$.
Now, $K$ satisfies (a) above. $N(K) \subset K$ and $N: K \to K$. If we let $h \in N(y)$, $y \in K$, $t_0, t_1 \in I$, $0 < \varepsilon \leq t_1 \leq t_2$ then for each $t \in I$ there exists a $g \in \mathbb{S}^\Phi$ such that
$|h(t)(\eta, \xi) - h(t_2)(\eta, \xi)|$
$\leq |h(t_2) - h(t_1)|y_0(\eta, \xi) - h(t_1)y_0(\eta, \xi)|$
$+ \int_0^{t_1} |T(t_2-s) - T(t_1-s)|B(s)(\eta, \xi) \, ds$
$+ \int_0^{t_1} |T(t_2-s) - T(t_1-s)|B(s)(\eta, \xi) \, ds$
$+ \int_0^{t_1} |T(t_2-s) - T(t_1-s)|B(s)(\eta, \xi) \, ds$
$+ \int_0^{t_1} |T(t_2-s) - T(t_1-s)|B(s)(\eta, \xi) \, ds$
$+ \int_t^{t_1} |T(t_2-s) - T(t_1-s)|B(s)(\eta, \xi) \, ds$
$+ \int_0^{t_1} |T(t_2-s) - T(t_1-s)|B(s)(\eta, \xi) \, ds$
$+ \sum_{0 < t_k < t} c_k |T(t_2-t_k) - T(t_1-t_k)|$ (14)
(14) tends to zero as $t_2 - t_1 \to 0$, for small $\varepsilon$. By the definition of $T(t)$, it is continuous, establishing the equicontinuity for the case when $t \neq t_k$.
When $t = t_k$ follows from the results in [4, 6, 7].
Using Arzela-Ascoli theorem, it is established that $N: K \to \mathcal{A}$, where $N$ is a multivalued map and $\mathcal{A}$ is a precompact set. Let
$0 < \varepsilon < t$ and fix $0 < t \leq \alpha$.
Now, for $y \in K$, define
$h(t)(\eta, \xi) = T(t)y_0(\eta, \xi) + T(\varepsilon)$
$+ \int_0^{t-\varepsilon} |T(t-s-\varepsilon)|B(s)(\eta, \xi) \, ds$
$+ \sum_{0 < t_k < t} T(t-s-t_k)J_k(y(t_k))(\eta, \xi) \, ds$
$+ \sum_{0 < t_k < t} T(t-s-t_k)J_k(y(t_k))(\eta, \xi) \, ds$
$+ \sum_{0 < t_k < t} T(t-s-t_k)J_k(y(t_k))(\eta, \xi) \, ds$
$+ \sum_{0 < t_k < t} T(t-s-t_k)J_k(y(t_k))(\eta, \xi) \, ds$
$+ \sum_{0 < t_k < t} T(t-s-t_k)J_k(y(t_k))(\eta, \xi) \, ds$
Now, by considering the operator
Since \( y_n \rightarrow y_\ast \), it follows that

\[
\begin{align*}
 h_* (t)(\eta, \xi) &= \eta, T(t) y_\ast(\xi) \\
 &- \sum_{0 < t_k < t} T(t - t_k) J_k (y_\ast(t_k))(\eta, \xi) \\
 &- \int_0^t T(t - s) B(y_\ast(s))(\eta, \xi) \, ds \\
 &- \int_0^t T(t - s) g_\ast(s) \, ds \\
 &\in \mathcal{S}^p.
\end{align*}
\]

for some \( g_\ast \in \mathcal{S}^p \). And the desired result is obtained.

IV. CONCLUSION
Having satisfied the conditions of Schaefer’s theorem, we conclude that \( N \) has a fixed point which is a solution of (3) respectively (4). If the operator \( B \) in equation (3) is zero, i.e., \( B = 0 \), then we obtain some results in [4, 6]. Hence, the result in this paper is a generalization of the results in the existing literature on impulsive quantum stochastic differential inclusion (equations).

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REFERENCES